Today's lesson is on what is considered by many to be the most important formula in Mathematics: Euler's formula. It goes as follows:

$$e^{ix}=\cos(x)+\sin(x)i$$

I won't type up the whole proof here, but basically when you substitute *ix* into the exponential function represented as its Taylor series, you get the Taylor series for the cosine function plus the Taylor series for the sine function times *i*. The Taylor series for these three functions are:

$$e^{x}=\sum_{n=0}^{\infty}\frac{x^{n}}{n!}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}\cdots$$

$$\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n)!}x^{2n}=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}\cdots$$

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+2)!}x^{2n+1}=1-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}\cdots$$

Each continuing on to infinity. You can see how Euler's formula works by substituting *ix* for *x* in the exponential function. Also note that the Taylor series for cosine and sine will only work if *x* is given in radians.

So, what can we use this for? The first thing is that we'll stop writing out complex numbers in polar form as $r(\cos(\theta)+\sin(\theta)i)$ and start writing then in the form of $re^{i\theta}$. From this it's also apparent how multiplying complex numbers in polar form is done by adding the arguments and multiplying the magnitudes.

Other than notational convention, Euler's formula obviously has more uses. The first we'll look at is raising one complex number to the power of another. First, we put the first number in polar form:

$$a+bi^{c+di}=(re^{i\theta})^{c+di}$$

Then we do some voodoo magic:

$$(e^{\ln(r)}e^{i\theta})^{c+di}=(e^{\ln(r)+i\theta})^{c+di}=e^{(\ln(r)+i\theta)(c+di)}$$

The key to this voodoo magic being that $e^{ln(r)} = r$, by definition. From here it's just distributive property and applying Euler's formula again.

$$e^{\ln(r)c-d\theta+(\ln(r)d+c\theta)i}=e^{\ln(r)c-d\theta}e^{(\ln(r)d+c\theta)i}=e^{\ln(r)c-d\theta}(\cos(\ln(r)d+c\theta)+\sin(\ln(r)d+c\theta)i)$$

There are technically infinitely many solutions because of the periodic nature of the sine and cosine functions, but we'll just be worrying about one solution per equation for the sake of example.

The formula for raising one complex number to another is very long and complicated to memorize, so it' easier to just remember the steps you go through than to know that long formula and substitute into it. So, for z^w:

Step 1: Put *z* in polar form using notation from Euler's formula

Step 2: Take |*z*| and replace it with $e^{ln(|z|)}$

Step 3: Represent *z* as $e^{ln(|z|)+Arg(z)i}$

Step 4: Using the property that $(a^b)^c = a^{bc}$ and distributive property, simplify.

Step 5: Simplify further using Euler's formula.

Woohoo! That's how you do that. There will be examples left as exercises at the bottom of this lesson.

Now, next thing we can do with this, which is significantly easier: We can calculate the exponential of a complex number!

$$e^{a+bi}=e^{a}e^{bi}=e^{a}(\cos(b)+\sin(b)i)$$

Fairly straightforward. Also as straightforward is the natural logarithm of a complex number:

$$\ln(ze^{i\theta})=\ln(e^{\ln(|z|)+i\theta})=\ln(|z|)+i\theta$$

If you need a logarithm of a different base of a complex number, you can use the following property:

$$\log_b(a) = \frac{\log_d(a)}{\log_d(b)}$$

In this case, *b* would be any value and *d* would be *e*.

If you know about Euler's formula, you should know what is probably the most important application of it:

$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0$$

This formula proves the transcendality of pi – that is, it is not the root of a nonzero polynomial equation with rational coefficients. That's fancy math speak that divides all numbers into two categories: algebraic and transcendental. Transcendental numbers can't be reduced into an algebraic number using everyday operations. Since *e* is transcendental and *i*, *1* and *0* aren't, pi has to be the number that is reducing *e* to *0* in this equation, making pi transcendental. I'm going off on a bit of a tangent here, but an important fact nontheless.

Alrighty then! I've got some time on my hands, so I think I'll type up the proof for Euler's formula after all:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \cdots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} \cdots$$

$$e^{ix} = 1 + ix + \frac{-x^2}{2!} + \frac{-x^3 i}{3!} + \frac{x^4}{4!} + \frac{x^5 i}{5!} \cdots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{x^3 i}{3!} + \frac{x^4}{4!} + \frac{x^5 i}{5!} \cdots$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\right) i$$

$$e^{ix} = \cos(x) + \sin(x) i$$

Hooray! That just about wraps up this lesson.

Optional exercises!

1. $\sqrt{2} e^{i\frac{3\pi}{4}} \cdot \sqrt{8} e^{i\frac{3\pi}{2}}$
2. $\sqrt{3} e^{i\frac{7\pi}{4}} \cdot \sqrt{3} e^{i\frac{5\pi}{6}}$
3. $(1+1i)^{(2+3i)}$
4. i^i
5. e^{2+5i}
6. e^{4-3i}
7. $\ln(2-3i)$